Basics of Complex Numbers (I)

1. **General**

   - $i \equiv \sqrt{-1}$, so $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ and then it starts over again.
   - Any complex number z can be written as the sum of a real part and an imaginary part:

     $$z = [\mathrm{Re}\, z] + i[\mathrm{Im}\, z] \ ,$$

     where the numbers or variables in the []'s are *real*. So $z = x + y\, i$ with x and y real is in this form but $w = 1/(a + b\, i)$ is *not* (see "Rationalizing" below). Thus, $\mathrm{Im}\, z = y$, but $\mathrm{Re}\, w \neq 1/a$.
   - **Complex Conjugate:** The complex conjugate of z, which is written as z^*, is found by changing the sign of every i in z:

     $$z^* = [\mathrm{Re}\, z] - i\, [\mathrm{Im}\, z] \qquad \text{so if } z = \frac{1}{a + b\, i}\,, \text{ then } z^* = \frac{1}{a - b\, i}\ .$$

     Note: There may be "hidden" i's in the variables; if a is a complex number, then $z^* = 1/(a^* - b\, i)$.
   - **Magnitude:** The magnitude squared of a complex number z is:

     $$zz^* \equiv |z|^2 = [\mathrm{Re}\, z]^2 - (i)^2[\mathrm{Im}\, z]^2 = [\mathrm{Re}\, z]^2 + [\mathrm{Im}\, z]^2 \geq 0\ ,$$

     where the last equality shows that the magnitude is positive (except when $z = 0$).
     Basic rule: if you need to make something real, multiply by its complex conjugate.

2. **Rationalizing:** We can apply this rule to "rationalize" a complex number such as $z = 1/(a + b\, i)$. Make the denominator real by multiplying by the complex conjugate on top and bottom:

   $$\frac{1}{a + b\, i} \cdot \frac{a - b\, i}{a - b\, i} = \frac{a - b\, i}{a^2 + b^2} = \frac{a}{a^2 + b^2} + i\frac{-b}{a^2 + b^2}$$

   so $\mathrm{Re}\, z = a/(a^2 + b^2)$ and $\mathrm{Im}\, z = -b/(a^2 + b^2)$.

3. **The Complex x–y Plane**

   - **Rectangular form:** Any complex number z can be uniquely represented as a point in the x–y plane, where the x–coordinate is $\mathrm{Re}\, z$ and the y–coordinate is $\mathrm{Im}\, z$ (see figure).
     - You can think of i as a unit vector in the "imaginary" (y) direction.
     - The magnitude of z is just the length of the vector from the origin.
   - **Polar form:** We can also write z in polar form as:

     $$z = r\, e^{i\theta} = r\, \cos\theta + i\, r\, \sin\theta\ ,$$

     where r and θ are real and equal to the length and angle of the vector.

- The complex conjugate of $z = r\,e^{i\theta}$ is $z^* = r\,e^{-i\theta}$.
- Thus the magnitude is $|z| = \sqrt{z\,z^*} = r$.
- Rationalizing:

$$\frac{1}{r\,e^{i\theta}} = \frac{1}{r}e^{-i\theta} \ .$$

4. **Multiplying Complex Numbers**

- Multiplication is distributive: $(a + b\,i) \times (c + d\,i) = (ac - bd) + i(ad + bc)$.
- In polar form, we multiply the r's and add the θ's: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1\,z_2 = r_1\,r_2\,e^{i(\theta_1+\theta_2)}$.

5. **Euler's Theorem and other Goodies:**

- From the polar form, we have:

$$e^{i\theta} = \cos\theta + i\,\sin\theta \ .$$

- Special values:

$$e^{2\pi i} = 1 \qquad e^{i\pi} = -1 \qquad e^{i\pi/2} = i \qquad e^{i\pi/4} = 1/\sqrt{2} + i/\sqrt{2}$$

- We can rewrite sin and cos:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad\qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \ ,$$

which can be very useful, since it is generally easier to work with exponentials then trigonometric functions.

- DeMoivre's Theorem:

$$z^n = (r\,e^{i\theta})^n = r^n\,e^{in\theta} = r^n[\cos(n\theta) + i\,\sin(n\theta)] \ .$$

We can also write the theorem in the form:

$$z^{1/n} = r^{1/n}[\cos(\theta/n) + i\,\sin(\theta/n)] \ ,$$

which is great for taking the square root, cube root, etc. of complex numbers!